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SOLVED PYQ

Paper: Advanced Mathematical Methods for Economics

Course: B. A.(Hons.) Economics II Year

Year: 2023-24

Unique Paper Code : 2272102303

Paper : ECON009: Advanced Mathematical methods for Economics

Course : B.A. (Hons.) Economics II Year

Duration : 3 Hours

Maximum Marks : 90

Semester–3

Dec./Jan.

2023–24

Each questions carries equal marks.

SECTION A

(*Attempt any 4 questions out of 6*)

Q. 1. (a) Let the rate of growth of output is given by the function : $Q(t) = 12t^{1/3}$. If $Q(0) = 20$, find the time path of output level. Also, find the total production during the initial three years. **5**

Ans. $\dot{Q}(t) = 12t^{1/3}, Q(0) = 20$

Time path of output $= \int 12t^{1/3} dt = 9t^{4/3} + Q(0)$, (orc) $= 9t^{4/3} + 20$

Total production during the initial three years

$$= \int_0^3 12t^{1/3} dt$$

$$= \left[9(t)^{4/3} + 20\right]_0^3$$

$$= \left[9(3)^{4/3} + 20\right] - \left[9(0)^{4/3} - 20\right]$$

$$= 9(3)^{4/3}$$

(b) Find the area of the region between the curves $y_1 = 3x^2 - 6x + 3$ and $y^2 = -2x^2 + 1$ within the interval [0, 2]. **5**

Ans.

$$y_1 = 3x^2 - 6x + 3 = 3(x-1)^2$$

$$y_2 = -2x^2 + 1$$

$\Rightarrow$

$$\int_0^2 \left(3x^2 - 6x + 3\right) - \left(-2x^2 + 1\right) dx$$

$$= \int_0^2 \left(3x^2 - 6x + 3 + 2x^2 - 1\right) dx$$

$$= \left[\frac{5x^3}{3}\right]_0^2 - \left[\frac{6x^2}{2}\right]_0^2 + [2x]_0^2$$

$$= 40/3 - 12 + 4 = \frac{16}{3} \text{ sq. unitst}$$

Q. 2. Solve the following difference equations. Also, determine whether the solution path is convergent or divergent and oscillatory or non-oscillatory.

(a) $x_t = -3x_{t-1} + 4, x_0 = 2$ 5

(b) $x_t = 0.5x_{t-1} + 3, x_0 = 5$ 5

Ans. (a) $x_t = -3x_{t-1} + 4$

Solution for this $x_t = 1 + (-3)^t(x_0 - 1)$

$$\left[\text{using } x_t = \frac{b}{1-a} + a^t\left(x_0 - \frac{b}{(1-a)}\right)\right]$$

Path is divergent and oscillating

Now,

using $x_0 = 2$

$$x_t = 1 + (-3)^t(2-1)$$
$$= 1 + (-3)^t$$

(b) $x_t = 6 + \left(\frac{1}{2}\right)^t (x_0 - 6)$ or $6 + (0.5)^t (x_0 - 6)$

Path is convergent and smooth

Now,

Using $x_0 = 5$

$$x_t = 6 + \left(\frac{1}{2}\right)^t (5-6)$$
$$= 6 - \left(\frac{1}{2}\right)^t$$

Q. 3. (a) The initial value of population of a country is 10^7. The birth rate is 0.04, death rate is 0.03 and 30,000 migrants arrive in the country every year. Write down the difference equation to represent this situation and solve it. Comment on its steady state. 6

Ans. Net growth rate = 0.04 – 0.03 = 0.01

Difference equation is

$$P_t = P_{t-1}[1 + 0.01] + 30,000$$
$$= 1.01(P_{t-1}) + 30,000$$
$$a = 1.01, b = 30,000$$

$$\therefore \quad \frac{b}{1-a} = \frac{30,000}{1-1.01} = -30,00,000$$

Also, $P_0 = 10 \times 10^6 = 1,00,00,000$

Sol. $P_t = (1.01)^t[1,00,00,000 - (-30,00,000)] + (-30,00,000)$

$$= (1.01)^t[1,30,00,000] + (-30,00,000)$$

Since, $a = 1.01 > 1 \therefore$ Diverges.

(b) A firm has current sales of ₹ 50,000 per month. The firm wants to embark upon a certain advertising campaign that will increase the sales by 2% every month (compounded continuously) over the period of 12 months of campaign. Find the total increase of sales because of the campaign. **4**

Ans. Let A denote the total increase of sales because of campign,

$$A = \left[\int_0^{12} 50{,}000e^{0.02t}\,dt\right] - 50{,}000 \times 12$$

$$= \frac{50{,}000}{0.02}\left[e^{0.02t}\right]_0^{12} - 6{,}00{,}000$$

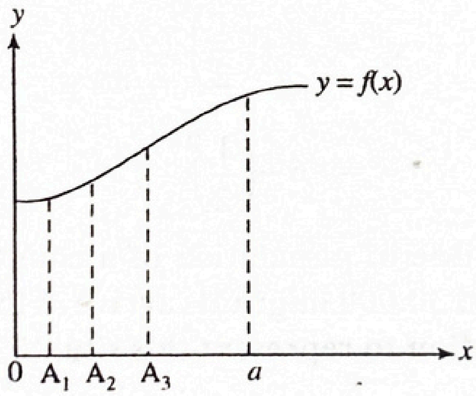
$$= 25{,}00{,}000\,(e^{0.024} - 1) - 6{,}00{,}000$$

Q. 4. (a) For a function $y = f(x)$ with the domain defined as $[0, a]$, where a is a positive constant. Find the area under the curve using Riemann integral. Give an approximated expression for the area. **6**

Ans. $y = f(x)$, Domain $[0, a]$

According to Reimann Integral,

$$\text{Area over } [0, a] = A_1 + A_2 + A_2 + \ldots + A_n$$
$$= (x_1 - x_0)f(0) + (x_2 - x_1)f(1) + (x_3 - x_2)f(2) + \ldots + (x_a - x_{a-1})f(a)$$



This sum will depend on f as well as on the subdivision and on the choice of xi's

$$\int_0^a f(x)\,dx = \sum_{x=0}^{n-1} f(\varepsilon_{ji})\,\Delta x_i$$

(b) For the following, evaluate $\frac{d}{dt}\int_{-t}^{t} \frac{1}{\sqrt{x^{4+1}}}\,dx$ and comment on the change in this integral value due to a unit change in t. **4**

Ans. Using the formula,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = f(b(t))\cdot b'(t) - f(a(t))\cdot a'(t)$$

$$\frac{d}{dt}\int_{-t}^{t}\frac{1}{\sqrt{x^4+1}}dx = \frac{1}{\sqrt{t^4+1}}-\frac{1}{\sqrt{(-t)^4+1}}(-1)$$

$$= \frac{2}{\sqrt{t^4+1}}>0$$

Q. 5. What is the present value of a continuous revenue flow lasting for *x* years at

(a) A constant rate of *R* dollars per year and discounted at the rate of *r* per year? 5

Ans.

$$\pi = \int_0^x \text{Re}^{-rt}\,dt = R\int_0^x e^{-rt}\,dt$$

$$= R\left[\frac{e^{-rt}}{-r}\right]_0^x$$

$$= \frac{R}{-r}\left[e^{-rx}-1\right]$$

$$= \frac{R}{r}\left[1-e^{-rx}\right]$$

(b) Find the present value in case of constant cash flow of: 5

(i) \$1450 per year, discounted at $r = 5\%$, $I = 2$ years

(ii) \$2460 per year, discounted at $r = 8\%$, $t = 3$ years

Ans. (i) $R = \$\,1450, \quad r = 5\%, \quad t = 2$ years

$$\pi = \frac{1450}{0.05}\left[1-e^{-0.05(2)}\right]$$

$$= 29{,}000\ (0.1052)$$

$$= 2050.8$$

(ii)

$$\pi = \frac{2460}{0.08}\left[1-e^{-0.08(3)}\right]$$

$$= 30{,}750\ [1-0.7866]$$

$$= 6562$$

Q. 6. Use the graphical method to solve the following LP problem. 6

Ans.

(a) Max $3x_1 + 5x_2$

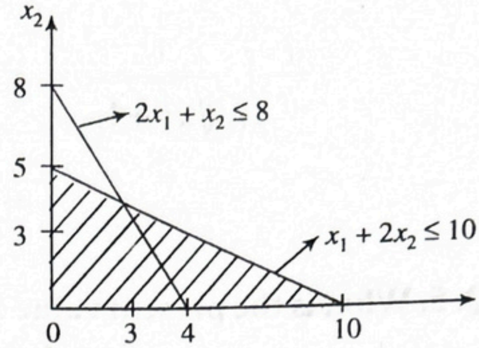
Subject to $x_1 + 2x_2 \le 10,$

$2x_1 + x_2 \le 8$

$2x_1 + 2x_2 \le 6$

(b) Write down the dual of the above problem. 4

Ans. (a) Max $3x_1 + 5x_2$



Subject to constraints

(3, 0), (0, 3), (0, 0)

$$\text{at } (3, 0);\ 3x_1 + 5x_2 = 9$$
$$\text{at } (0, 3);\ 3x_1 + 5x_2 = 15$$
$$\text{at } (0, 0);\ 3x_1 + 5x_2 = 0$$

$\therefore$ At (0, 3) the function is maximized.

(b) Min $10\,V_1 + 8V_2 + 6V_3$

Subject to,

$$V_1 + 2V_2 + 2V_3 \geq 3$$
$$2V_1 + V_2 + 2V_3 \geq 5$$

SECTION B

(Attempt any 3 questions out of 4)

Q. 7. (a) The value of a mqachine depreciates over time according to the relation

$$\frac{dV}{dx} = 750\,e^{-0.03t}$$

where V denotes the value of machine in rupees and t denotes time in years. Find depreciation in a period of 5 years. **4**

Ans. Value of Machine is

$$V = \int \frac{dV}{dt}\,dt = \int 750\,e^{-0.03\,t}\,dt$$
$$= -25{,}000\,e^{-0.03\,t}\,dt + C$$

$$\text{Value of New machine} = V_0 = -25{,}000 + C\ (t = 0)$$

$$\text{Value of New Machine at end of 5 years} = V_5 = -25{,}000\,e^{-0.15} + C$$

$$\text{Depreciation} = V_0 - V_5$$
$$= -25{,}000 - [-25{,}000\,e^{-0.15}]$$
$$= -25{,}000\,[1 - e^{-0.15}]$$
$$= -25{,}000\left[1 - \frac{1}{e^{0.15}}\right]$$

(b) A firm uses inputs L and K to produce a target level of output $Q = LK$, where L and K represent Labour and Capital respectively. The prices per unit of L and K are w and r respectively. Solve the following minimization problem:

$$\textbf{Minimise } C(L, K) = wL + rK$$
$$\textbf{subject to } Q = LK$$

Find the cost-minimizing level of inputs, L^* and K^*. Comment on the relation between these optimal values and-the level of output Q.

Ans. $L = wL + rK + \lambda\,(Q - LK) \qquad \because LK = Q$

$$\frac{\partial L}{\partial L} = w - \lambda K = 0,$$

$$\frac{\partial L}{\partial \lambda} = \text{Q} - \text{LK}$$

$$\frac{\partial L}{\partial K} = r - \lambda L = 0$$

$$L^* = \sqrt{\frac{rQ}{w}},$$

$$K^* = \sqrt{\frac{wQ}{r}}$$

$$C^* = wL^* + rK^* = 2\sqrt{wrQ}$$

$$\frac{rC^*}{rQ} = \lambda^* = \sqrt{\frac{wr}{Q}} > 0$$

Q. 8. (a) Evaluate the following definite integral $\int_0^4 f(x)\,dx$

when $f(x) = \begin{cases} \sqrt{4x+1}; & 0 \le x \le 1 \\ x^2 + 2x + 3; & 1 \le x \le 4 \end{cases}$ 4

Ans.

$$\int_0^4 f(x)\,dx = \int_0^1 \sqrt{4x+1}\,dx + \int_1^4 (x^2 + 2x + 3)\,dx$$

$$= \left[\frac{(4x+1)^{3/2}}{3/2} \times \frac{1}{4}\right]_0^1 + \left[\frac{x^3}{3} + \frac{2x^2}{2} + 3x\right]_1^4$$

$$= \left[\frac{1}{4} \times \frac{2}{3} \times (5)^{3/2} - \frac{1}{4} \times \frac{2}{3} \times (1)^{3/2}\right]$$

$$+ \left[\frac{64}{3} + \frac{32}{3} + 12 - \frac{1}{3} - \frac{2}{3} - 3\right]$$

$$= \frac{1}{6}(\sqrt{5})^3 + \frac{113}{6}$$

(b) Find the differential equation of the family of circles passing through the origin and having centre on the y-axis. 6

Ans. $$x^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad \frac{x^2 + y^2}{2y} = a \qquad \text{...(1)}$$

Differential equation (1) w.r.t. x

$$\Rightarrow \quad 4xy + 4y^2\left(\frac{dy}{dx}\right) - 2(x^2 + y^2)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Q. 9. (a) Fod the following equation $\frac{dy}{dt} = 5y - 5$ where $y(0) = 5$. Show that $y_t = 2e^{st} + 1$ is a solution to the above differential equation and comment on its equilibrium state? **4**

Ans. $$y(t) = 2e^{5t} + 1, \frac{dy}{dt} = 10e^{5t}$$

Now, $$10e^{5t} = 5(2e^{5t} + 1) - 5 = 10e^{5t}$$

Since $Y(t)$ satisfigs the given differentuial equation, solution;

$$\frac{dy}{dt} = \dot{Y} = 5Y - 5 \qquad \Rightarrow F'(\dot{Y}) = 5 > 0$$

(Implying that equilibrium is unstable)

Hence, $Y(t)$ does not converge to steady state.

Also, $$\lim_{t \to \infty} Y(t) = \infty;$$

Implying that it does not vonverge to steady state.

(b) For the following National Income Accounting problem:

$$Y_0 = 1500$$

$$I_0 = 50$$

and $$C_0 = 90 + 0.10\, Y_{t-1}$$

National Income Accounting Equation is given by $Y = C + I$. Find the time path of the national income (Y_t) at time t. Also comment on the stability of this time path. **6**

Ans. $$Y_t = 90 + 0.10\, Y_{t-1} + 50$$

$$Y_t = 140 + 0.10\, Y_{t-1}$$

$$Y_t = \left(1500 - \frac{140}{1-0.10}\right)(0.10)^t + \frac{140}{1-0.10}$$

$$= 1344\,(0.10)^t + 156 \text{ (using } \quad = 1500)$$

($a = 0.10$), $a > 0$ and $|a| < 1$; Implying that time path y_t is cor rgent.

Q. 10. (a) Determine the solutions of the difference equation and characterise the time path

$$2x_t + x_{t-1} + 2 = 0; \qquad x_0 = -1$$ 5

Ans.
$$2x_t + x_{t-1} + 2 = 0$$
$$x_t = \frac{-1}{2} x_{t-1} - \frac{2}{2}$$
$$= \frac{-1}{2} x_{t-1} - 1$$

Solution to difference equation:

$$x_t = \left(\frac{-1}{2}\right)^t \left[-1 - \frac{(-1)}{1+1/2}\right] + \frac{(-1)}{1+1/2}$$
$$x_t = \left(\frac{-1}{2}\right)^t \left[-1 + \frac{2}{3}\right] + \left(\frac{-2}{3}\right)$$
$$= \left(\frac{-1}{2}\right)^t \left(\frac{-1}{3}\right) - \frac{2}{3}$$
$$|a| = |-1/2| = |1/2| < 1$$

Solution converges to equilibrium state, hence equation is stable.

(b) Solve the following integral

$$\int_e^6 \left(\frac{1}{1+x} + x\right) dx$$ 5

Ans.
$$\int_e^6 \left(\frac{1}{1+x} + x\right) dx = \left[\ln|1+x| + \frac{x^2}{2}\right]_e^6$$
$$= \ln 7 + \frac{36}{2} - \ln|1+e| - \frac{e^2}{2}$$
$$= \ln\left|\frac{7}{1+e}\right| + \frac{36}{2} - \frac{e^2}{2}$$

SECTION C

(Attempt any 2 questions out of 3)

Q. 11. A firm produces two commodities A and B. The firm has three factories that jointly produce both commodities in the amounts per hour given in the following table

	Factory A	Factory B	Factory C
Commodity A	10	20	20
Commodity B	20	10	20

The firm receives an order for 300 units of A and 500 units of B. The cost per hour of running factories 1,2 and 3 are respectively 10,000, 8,000 and 11,000.

(*a*) Let y_1, y_2 and y_3 respectively denote the number of hours for which the three factories are used. Write down the linear programming problem of minimising the costs of fulfilling the order and find its solution. **3**

Ans. Min. $10{,}000\, y_1 + 8{,}000\, y_2 + 11{,}000\, y_3$

Subject to:

$$10\, y_1 + 20\, y_2 + 20\, y_3 \geq 300$$
$$20\, y_1 + 10\, y_2 + 20\, y^3 \geq 500$$
$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0$$

At

$$x_1^* = 100$$
$$x_2^* = 450$$
$$10\, x_1^* + 20\, x_2^* = 10{,}000 \quad \Rightarrow \quad y_1^* > 0$$
$$20\, x_1^* + 10\, x_2^* = 6{,}500 < 8{,}000 \quad \Rightarrow \quad y_2^* = 0$$
$$20\, x_1^* + 20\, x_2^* = 11{,}000 \quad \Rightarrow \quad y_3^* > 0$$

Incorporating this in primal constraints,

$$10\, y_1 + 20\,(0) + 20\, y_3 = 300$$
$$20\, y_1 + 10\,(0) + 20\, y_3 = 500$$
$$- \quad - \quad - \quad -$$
$$-10\, y_1 = -200$$
$$y_1^* = 20,$$
$$y_3^* = 5$$

$\therefore$ Solution to primal

$$y_1^*, y_2^*, y_3^* = (20, 0, 5)$$

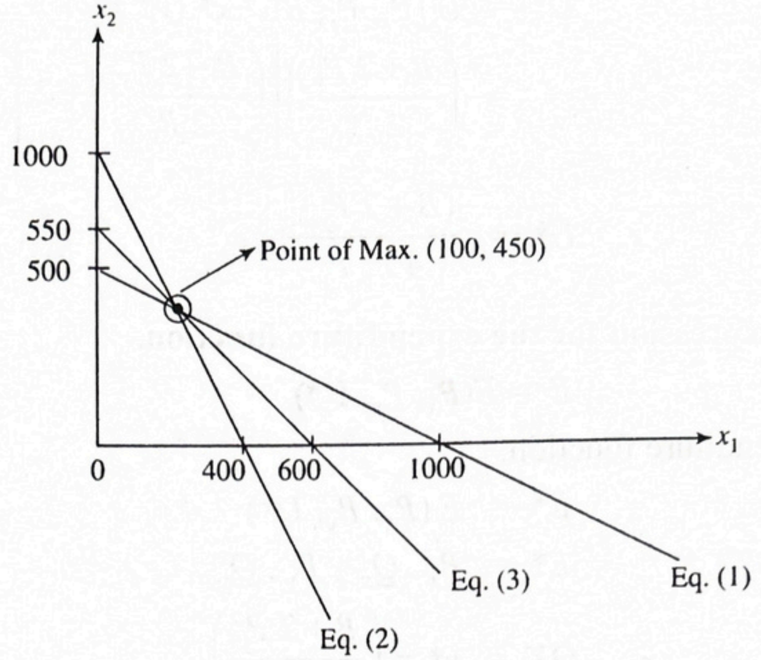
(*b*) Write down the dual problem of part (a) and find the solution. **4**

Ans. Max $300\, x_1 + 500\, x_2$

Subject to

$$10\, x_1 + 20\, x_2 \leq 10{,}000 \quad \text{...(1)}$$
$$20\, x_1 + 10\, x_2 \leq 8{,}000 \quad \text{...(2)}$$
$$20\, x_1 + 20\, x_2 \leq 11{,}000 \quad \text{...(3)}$$

$$\text{Max. Value} = 300\,(100) + 500\,(450)$$
$$= 30{,}000 + 2{,}25{,}000$$
$$= 2{,}55{,}000$$



(c) By how much will the minimum cost of production increase if the cost per hour in factory 1 increase by 100? **3**

Ans. Increase in cost = 100 (20) = 2000

$$\Delta = y_1^* (100) + y_2^* (0) + y_3^* (0)$$

Q. 12. Consider a consumer with the cost function $U(x, y) = x(y + 2)$, who faces a budget constraint of B, and prices of good x and good y are P_x and P_y respectively.

(a) From the first-order conditions, determine the expression for the demand function. **3**

Ans.

$$Z = x(y+2) - \lambda (P_x \cdot x + p_y \cdot y - B)$$

$$\frac{\partial Z}{\partial x} = (y+2) - P_x . \lambda = 0$$

$$\frac{\partial Z}{\partial y} = x - P_y . \lambda = 0$$

$$\frac{\partial Z}{\partial \lambda} = P_x . x + P_y . y = B$$

$$x = P_y . \left(\frac{y+2}{P_x}\right)$$

$$y^* = \left(\frac{B - 2P_y}{2P_y}\right)$$

$$x^* = \left(\frac{B + 2P_y}{2P_x}\right)$$

(b) Find an expression for the indirect utility function.

$$\boldsymbol{U^* = U(P_x, P_y, B)}$$ **2**

Ans.

$$U^* = U(P_x, P_y, B) = x^* (y^* + 2)$$

$$= \left(\frac{B+2P_y}{2P_x}\right)\left[\left(\frac{B-2P_y}{2P_y}\right)+2\right]$$

$$U^* = \frac{(B+2P_y)^2}{4P_x P_y}$$

(c) Find an expression for the expenditure function.

$$\boldsymbol{E^* = E(P_x, P_y, U^*)}$$ **2**

Ans. The expenditure function:

$$E^* = E(P_x, P_y, U^*)$$

$$E^* = P_x . Q_x + P_y . Q_y$$

Since $QX = x^* = \left(\frac{B+2P_y}{2P_x}\right)$

and $Q_y = y^* = \frac{B-2P_y}{2P_y}$

$\therefore$ $$E^* = P_x \cdot \left(\frac{B+2P_y}{2P_x}\right) + P_y\left(\frac{B-2P_y}{2P_y}\right)$$

$$= \left(\frac{B+2P_y}{2}\right)+\left(\frac{B-2P_y}{2}\right)$$

$$= \frac{1}{2}\left[B+2\not{P_y}+B-2\not{P_y}\right]$$

$$= \frac{1}{2}[2B]$$

$$= B$$

(d) Show that if the problem changes to

Min $P_x x + P_y y$ subject to $x(y+2) = U^*$

Show that x and y that solve the minimisation problem are equal to partial derivatives of the expenditure function. **3**

Ans.

$$\frac{\partial L}{\partial x} = P_x - \lambda(y+2) = 0$$

$$P_x = \lambda(y=2) \Rightarrow 1 = \frac{P_x}{y+2} \quad ...(1)$$

$$\frac{\partial L}{\partial y} = P_y - \lambda x = 0$$

$$P_y = \lambda x \Rightarrow 1 = \frac{P_y}{x} \quad ...(2)$$

Equating (1) and (2)

$$\frac{P_x}{y+2} = \frac{P_y}{x}$$

$$\Rightarrow \quad P_x . x = P_y (y+2)$$

Using constraints $x (y + 2) = U$

$$y = \sqrt{\frac{U^* P_x}{P_y}} - 2$$

Substituting y back into $x = \dfrac{P_y (y+2)}{P_x}$

$$x = \sqrt{\frac{U^* P_y}{P_x}}$$

And expenditure function:

$$E (P_x, P_y, U^*) = P_x . x + P_y . y$$

$$\frac{\partial E}{\partial P_x} = x^*$$

$$\frac{\partial E}{\partial P_y} = y^*$$

$$\therefore \quad x^* = \sqrt{\frac{U^* P_y}{P_x}}$$

and

$$y^* = \sqrt{\frac{U^* P_x}{P_y}} - 2$$

Thus, these are equal to the partial derivatives of the expenditure function.

Q. 13. An individual purchases quantities a, b, c of three different commodities whose prices are p, q, r respectively. The consumer's exogenous income given is M where $M > 2p$. The utility function is defined as $U(a, b, c) = a + \ln (bc)$.

(a) Using the Lagrangean method, find the consumer's demand for each good as function of prices and income. **4**

Ans.

$$L = a + \ln (bc) - \lambda (Pa + bq + Cr - M)$$

$$\frac{\partial L}{\partial a} = 1 - \lambda P = 0$$

$$\frac{\partial L}{\partial b} = \frac{C}{bc} - \lambda q = 0$$

$$\frac{\partial L}{\partial C} = \frac{b}{bC} - \lambda r = 0$$

Sol.
$$a^* = \frac{M}{p} - 2 \quad b^* = \frac{p}{q}$$

$$c^* = \frac{P}{r}$$

(b) Show that the ratio between marginal utility of a commodity and its price per unit must be same for all the commodities. **3**

Ans. Equating, λ's value from above FOC's gives

$$\lambda = \frac{1}{p} = \frac{1/b}{q} = \frac{1/C}{r}$$

This shows that ratio of MU and its price per unit are equal for all commodities *a*, *b* and *c*.

(c) Show that the expenditure on second and third good are always equal. **3**

Ans.
$$b^* = \frac{P}{q}$$

Expenditure on second good, $b \cdot q = p$

$$c^* = \frac{P}{r} \text{ (from above)}$$

Expenditure on third good, $C \cdot r = P$

Hence, This shows both are equal.

□□□